

February 1, 2008

BLACK HOLE UNIQUENESS THEOREMS*

Pawel O. Mazur[†]

Department of Physics and Astronomy,
University of South Carolina, Columbia, S. C. 29208, U. S. A.

Abstract

I review the *black hole uniqueness theorem* and the *no hair theorems* established for *physical* black hole stationary states by the early 80'. This review presents the original and decisive work of Carter, Robinson, Mazur and Bunting on the problem of no bifurcation and uniqueness of *physical* black holes. Its original version was written only few years after my proof of the Kerr-Newman et al. black hole uniqueness theorem²³ has appeared in print. The proof of the black hole uniqueness theorem relies heavily on the positivity properties of nonlinear sigma models on the Riemannian non-compact symmetric spaces with negative sectional curvature. It is hoped that the first hand description of the original developments leading to our current understanding of the black hole uniqueness will be found useful to all interested in the subject.

*This review is based on my plenary lecture at GR11 in Stockholm, July 1986. The plenary lecture was published in *Proceedings of the 11th International Conference on General Relativity and Gravitation*, ed. M. A. H. MacCallum, Cambridge University Press, Cambridge 1987, pp. 130-157.

[†]E-mail address: mazur@mail.psc.sc.edu

I. Introduction

It is well known that spherical gravitational collapse produces a black hole (Oppenheimer and Snyder, 1939).³¹ There are reasons to believe that this is true for collapse with small deviations from spherical symmetry (Doroshkevich, Zel'dovich and Novikov, 1966)⁸ and it was conjectured by Penrose (1969)³² that no ‘naked singularities’ can occur during complete gravitational collapse (Cosmic Censorship Conjecture). The Cosmic Censorship Conjecture is also the fundamental and not yet proved assumption we make when studying black hole equilibrium states. The considerable body of work on perturbations of stationary black holes brought us the presently accepted picture of gravitational collapse (Regge and Wheeler 1957; Visheshwara 1970; Ipser 1971; Price 1972; Press and Teukolsky 1973; Wald 1973, and others).^{35,40,19,33,34,39} One would expect that a black hole formed by gravitational collapse would settle down to a stationary equilibrium state. Properties of equilibrium states were extensively studied in the late sixties and early seventies (Carter, 1971, 1973; Bardeen, Carter and Hawking, 1973; Hawking and Ellis, 1973; Hawking, 1973).^{5,6,1,18,17} Theorems of Israel, Carter, Hawking and Robinson obtained between 1967 and 1975 gave proof of the remarkable result that Kerr (1963)²² black holes are the only possible stationary vacuum black holes. The first black hole uniqueness theorem came as a surprise when Israel (1967, 1968)^{20,21} proved that a static, topologically spherical black hole is described by the Schwarzschild or the Reissner-Nördström solutions. No hair and uniqueness theorems for a stationary axisymmetric, topologically spherical black hole were obtained by Carter (1971, 1973),^{5,6} Robinson (1974, 1975),^{36,37} Mazur (1982)^{23,24,26,27} (and independently by Bunting (1983)⁴) using the Ernst (1968)¹⁰ and Geroch (1971)¹² formulation of the Einstein equations for a stationary and axisymmetric gravitational field equations with a nice positivity property.

The basic assumptions made in a proof of uniqueness theorems were justified by Hawking (1973)¹⁷ who demonstrated that a stationary black hole must be static or axisymmetric and the horizon has a spherical topology. Also, Hajicek (1973)¹³ has shown that the outer boundary of the ergosphere must always intersect the event horizon. All these results depend on the validity of the Cosmic Censorship Hypothesis. Causality of a part of spacetime outside the event horizon is also assumed in a proof of uniqueness theorems.

In this lecture I describe recent work on black hole uniqueness theorems,

reformulating the problem of uniqueness of the Kerr-Newman et al. black hole solutions as a problem in the harmonic map theory. I would like to refer the reader to the excellent review article by Carter (1979)⁷ for a more detailed account of basic assumptions made in a proof of uniqueness theorems.

II. The Einstein-Maxwell Equations for Spacetimes with One Killing Vector

The Ernst-Geroch formulation of the Einstein-Maxwell equations for the stationary axisymmetric case can be carried out with respect to one of the two existing Killing vectors. It is crucial for the proof of black hole uniqueness theorems to have such a formulation which leads to the harmonic maps between two Riemannian spaces, because only then may one hope to have a global divergence identity with the required positivity property. One can define, therefore, the Ernst potential with respect to the stationary Killing vector $\partial/\partial t$ or with respect to the axial symmetry Killing vector $\partial/\partial\phi$. However, as we will see below, it is the second choice which leads to a Riemannian metric on the image space of the Ernst potential because the norm $X = (\partial/\partial\phi, \partial/\partial\phi)$ is everywhere positive and vanishes only on the symmetry axis. X is positive because we assume that the black hole spacetimes are causal in the domain of outer communications (Carter, 1979).⁷ What happens when we use the locally timelike Killing vector $\partial/\partial t$? First of all, we notice that the norm $-V = (\partial/\partial t, \partial/\partial t)$ of $\partial/\partial t$ changes sign on the ergosurface of rotating black holes. This means that the signature of the metric on the image space \mathbf{N} of the Ernst potentials is changing as we cross the ergosurface of the black hole. The metric on \mathbf{N} is pseudoriemannian inside the ergosphere. For this reason the global divergence identity is losing its nice positivity property and besides the difficulties with deriving boundary conditions for black holes, in this case, we cannot apply it in the proof of the black hole uniqueness conjecture!

It may be useful to give a short presentation of a reduction of E-M equations from 4-dim to 3-dim using a covariant Geroch formulation for the case with one Killing vector $\xi = \xi^a \partial_a$. Let $(\mathcal{M}, g_{ab}), F_{ab}$ be solutions to the Einstein-Maxwell equations

$$R_{ab} - \frac{1}{2}Rg_{ab} = 8\pi T_{ab}, \quad (1)$$

$$D_b F^{ab} = 0, \quad (2)$$

$$D_{[a} F_{bc]} = 0, \quad (3)$$

where

$$T_{ab} = (4\pi)^{-1} \left(F_a^c F_{bc} - \frac{1}{4} g_{ab} F_{cd} F^{cd} \right), \quad (4)$$

and F_{ab} is the electromagnetic strength tensor.

Assume that (g_{ab}, F_{ab}) are invariant with respect to a one-parameter isometry group generated by the Killing vector field $\xi = \xi^a \partial_a$, i. e., $\mathcal{L}_\xi g = \mathcal{L}_\xi F = 0$. Our convention for the signature of g_{ab} is $(-+++)$. The projection of the E-M equation on the space of orbits of ξ is straightforward (we follow Geroch formulation and notation (Geroch 1971)¹²). The metric on the space of orbits M is $h_{ab} = g_{ab} - \lambda^{-1} \xi_a \xi_b$, where $\lambda = \xi_a \xi^a$. At this point it is convenient to notice that when ξ is timelike, then h_{ab} is Riemannian; on the other hand, if ξ is spacelike, as is the case with $\partial/\partial t$ Killing vector inside the ergosphere of a black hole, then h_{ab} is pseudoriemannian and the reduced field equations are hyperbolic in this region! One can also define a twist vector $\tilde{\omega}_a$ which is non-vanishing if ξ is not hypersurface orthogonal Killing vector, $\tilde{\omega}_a = \epsilon_{abcd} \xi^b D^c \xi^d$. In the vacuum case knowing $(M, h_{ab}, \lambda, \tilde{\omega}_a)$ one can reconstruct (\mathcal{M}, g_{ab}) completely. In the electrovacuum case one needs more information, i. e., we have to know the ‘electric’ and ‘magnetic’ fields defined as follows: $E^a = -F_{ab} \xi^b$, $B_a = -{}^* F_{ab} \xi^b$, which are automatically defined on M , i. e., $E_a \xi^a = B_a \xi^a = 0$. Knowing E_a, B_a one can reconstruct F_{ab} :

$$F_{ab} = \lambda^{-1} (2\xi_{[a} E_{b]} - \epsilon_{abcd} \xi^c B^d). \quad (5)$$

It can be easily seen (Carter, 1972)⁶ that (2), (3) and the Killing equations on (g, F) imply the existence of ‘electric’ and ‘magnetic’ potentials E and B in a simply connected region of M , i. e., $E_a = \nabla_a E$, $B_a = \nabla_a B$. The basic equations for spacetimes with one Killing vector ξ which we apply in derivation of the reduced E-M equations are the following (we present them here for the sake of completeness; it can be seen that using them and the E-M equations, one arrives easily at the final Ernst form of the field equations. In the following, we will only indicate the main steps in the derivation of the reduced equations.)

$$D_a \xi_b = \frac{1}{2} \lambda^{-1} \epsilon_{abcd} \xi^c \tilde{\omega}^d + \lambda^{-1} \xi_{[b} \nabla_{a]} \lambda, \quad (6)$$

$$D_a D_b \xi_c = \xi^d R_{dabc}, \quad (7)$$

$$\nabla_{[a} \tilde{\omega}_{b]} = -\epsilon_{abcd} \xi^c R_e^d \xi^e, \quad (8)$$

$$\nabla_a \tilde{\omega}^a = \frac{3}{2} \lambda^{-1} \tilde{\omega}^a \nabla_a \lambda, \quad (9)$$

$$\nabla_a \nabla^a \lambda = \frac{1}{2} \lambda^{-1} \nabla_a \lambda \nabla^a \lambda - \lambda^{-1} \tilde{\omega}_a \tilde{\omega}^a - 2 R_{ab} \xi^a \xi^b, \quad (10)$$

$${}^{(3)}R_{ab} = \frac{1}{2} \lambda^{-2} (\tilde{\omega}_a \tilde{\omega}_b - h_{ab} \tilde{\omega}_c \tilde{\omega}^c) + \frac{1}{2} \lambda^{-1} \nabla_a \nabla_b \lambda - \frac{1}{4} \lambda^{-2} \nabla_a \lambda \nabla_b \lambda + h_a^c h_b^d R_{cd}, \quad (11)$$

where ${}^{(3)}R_{ab}$ is the Ricci tensor of h_{ab} . Using the field equations (1), (4) and equations (7), (10), we arrive at

$$\nabla_{[a} \tilde{\omega}_{b]} = -4 E_{[a} B_{b]}. \quad (12)$$

It is easy to see that we can define a curl-less ‘twist’ vector

$$\omega_a = -\tilde{\omega}_a + 2 (B E_a - E B_a), \quad (13)$$

i. e., $\nabla_{[a} \omega_{b]} = 0$. In a simply connected region one can define, therefore, a potential $\omega : \omega_a = \nabla_a \omega$. It is easy to check that if we conformally rescale the metric h_{ab} using a scale factor depending on λ we can cancel the second derivatives of λ in the expression for the Ricci tensor on the manifold of orbits. The required conformal transformation has the form: $h_{ab} = \lambda^{-1} \gamma_{ab}$.

Introducing the complex Ernst potentials ϵ and ψ as follows

$$\epsilon = -\lambda - \psi \bar{\psi} + i\omega, \quad \psi = E + iB, \quad (14)$$

and taking all covariant derivatives ∇_a with respect to γ_{ab} , we can write equations (11), (12) and (13) in a nice symmetric form of 3-dim Einstein equations coupled to the Ernst equations:

$$-\lambda \nabla_a \nabla^a \epsilon = (\nabla_a \epsilon + 2\bar{\psi} \nabla_a \psi) \nabla^a \epsilon, \quad (15)$$

$$-\lambda \nabla_a \nabla^a \psi = (\nabla_a \epsilon + 2\bar{\psi} \nabla_a \psi) \nabla^a \psi, \quad (16)$$

$$R_{ab}(\gamma) = \frac{1}{2} \lambda^{-2} [\nabla_{(a} \epsilon \nabla_{b)} \bar{\epsilon} + 2\psi \nabla_{(a} \epsilon \nabla_{b)} \bar{\psi} + 2\bar{\psi} \nabla_{(a} \bar{\epsilon} \nabla_{b)} \psi + 4\lambda \nabla_{(a} \psi \nabla_{b)} \bar{\psi}] . \quad (17)$$

Now it can be easily seen that equations (15), (16) and (17) can be derived from the action principle of the 3-dim gravity coupled to a harmonic map model

$$S = \int |\gamma|^{\frac{1}{2}} d^3x (R - L), \quad (18)$$

where R is the Ricci scalar of the metric γ_{ab} and L is given by

$$L = \frac{1}{2} \lambda^{-2} [|\nabla \epsilon + 2\bar{\psi} \nabla \psi|^2 + 4|\nabla \psi|^2]. \quad (19)$$

L defines an ‘energy’ integral for a harmonic map between two spaces (M, γ_{ab}) and (N, H_{AB}) , i. e., a map between the space of orbits M of ξ with its conformal metric γ_{ab} and the space N of Ernst’s potentials $\Phi^A = (\epsilon, \psi)$ with the metric H_{AB} as defined by L . At this point let me recall the definition of the *harmonic map*. I will give here a definition using coordinate systems on two spaces M and N , but one can easily see that it is a coordinate invariant statement.

Definition 1 *Let (M, γ) and (N, H) be two (pseudo)Riemannian manifolds with metrics γ and H respectively. A map $g : M \rightarrow N$ which in local coordinate systems x^a and Φ^A on M and N , respectively, can be written as: $\Phi^A = g^A(x^a)$ is called a harmonic map if the energy functional*

$$I[g] = \frac{1}{2} \int_M |\gamma|^{\frac{1}{2}} dx \gamma^{ab} H_{AB}(\Phi) \nabla_a \Phi^A \nabla_b \Phi^B, \quad (20)$$

is stable under small deformations of g , i. e., if the Lagrange-Euler equations for the functional $I[g]$ are satisfied (Bochner, 1940; Fuller, 1954; Eells and Sampson, 1964).^{2,11,9} N is called the image space of the harmonic map g .

The Lagrangian (19) defines the hermitian metric on the image space N of the Ernst potentials ϵ and ψ

$$ds^2 = \frac{1}{2} (\text{Re } \epsilon + \psi \bar{\psi})^{-2} [|d\epsilon + 2\bar{\psi}d\psi|^2 - 4(\text{Re } \epsilon + \psi \bar{\psi}) d\psi d\bar{\psi}]. \quad (21)$$

It is quite interesting to study properties of this metric. The metric (21) is Riemannian only when $\lambda = -(\text{Re } \epsilon + \psi \bar{\psi}) > 0$; otherwise it is a pseudoriemannian metric. It will be demonstrated later that the image space N is in fact the symmetric homogeneous space $N = SU(2, 1)/S(U(2) \times U(1))$ when $\lambda > 0$, and $N = SU(2, 1)/S(U(1, 1) \times U(1))$ when $\lambda < 0$ (Mazur, 1982a,b, 1984a,b).^{23,24,26,27} Harmonic maps onto homogeneous spaces G/H are known in physics as nonlinear sigma models. This peculiar sigma model property of the E-M field equations with one Killing vector proved to be crucial in proving black hole uniqueness theorems for stationary and axisymmetric black holes. Because we are interested in stationary and axisymmetric solutions we have to consider further reduction of the E-M field equations. We assume that there is another Killing vector $\eta = \eta^a \partial_a$ commuting with the previous Killing vector $\xi = \xi^a \partial_a$, i. e., $[\xi, \eta] = 0$. Subsequent $3 \rightarrow 2$ reduction from the space of orbits of ξ to the space of orbits of ξ and η is straightforward. The Ernst potentials ϵ and ψ defined with respect to ξ will be well defined on the space of orbits $M' = \mathcal{M}/G_2$, where G_2 is the abelian isometry group generated by ξ and η , only if the action of G_2 on \mathcal{M} has the orthogonal transitivity property. It means that there exists a family of 2-surfaces orthogonal to 2-surfaces of transitivity of G_2 . The dimensional $4 \rightarrow 2$ reduction of the E-M equations from 4 to 2 dimensions can be carried out if the reduction does not depend on the order of the reduction in two steps. We first consider $4 \rightarrow 3$ reduction with respect to ξ and later $3 \rightarrow 2$ reduction with respect to η . The result of this reduction should be the same if we change the order of subsequent steps, i. e., we first reduce with respect to η and later with respect to ξ . The consistency conditions for the $4 \rightarrow 2$ reduction are, of course, the vanishing of Lie derivatives of ϵ and ψ with respect to the second Killing vector, i. e., $\mathcal{L}_\eta \epsilon = \mathcal{L}_\eta \psi = 0$, where ϵ and ψ are defined with respect to ξ . Analogous conditions should be satisfied for the reversed order of the two steps of reduction; they are equivalent to the Frobenius integrability condition for the orthogonal transitivity of G_2 . As a result of reduction from 4 to 2 dimensions, we will obtain a 2-dimensional gravity. Because any 2-dimensional manifold is conformally flat and a nonlinear sigma model is

conformally invariant in two dimensions, we observe that nonlinear sigma model equations will decouple from the two-dimensional gravity. It means that we can solve first nonlinear sigma model equations on the 2-dimensional manifold of orbits M' choosing the flat metric on M' . The remaining 2-dim Einstein equations will determine the metric $g_{\mu\nu} = e^{2\Gamma}\delta_{\mu\nu}$, $\mu = 1, 2$, i. e., its conformal scale Γ on M' (we have chosen a ‘conformal gauge’ on M'). Once the solution to nonlinear sigma model equations is known we can determine the metric $g_{\mu\nu}$ on M' and we can reconstruct the four-dimensional spacetime (\mathcal{M}, g_{ab}) completely. One can see, therefore, that the problem of uniqueness of black hole solutions of the Einstein-Maxwell equations can be studied as a two dimensional boundary value problem for the nonlinear sigma model equations.

III. Black Hole Boundary Conditions

To this end let me present the black hole boundary value problem. We refer the reader to the more detailed derivation of these conditions discussed by Carter (Carter, 1973, 1979).^{6,7} We will follow Carter’s notation. Thus if $\xi = \partial/\partial\phi$ we denote $\lambda = X$ and $\omega = Y$. The determinant of the matrix of scalar products of two Killing vectors $\xi = \partial/\partial\phi$, $\eta = \partial/\partial t$ defines the real scalar field $\rho : -\rho^2 = (\xi, \xi)(\eta, \eta) - (\xi, \eta)^2$. The scalar field ρ vanishes on the symmetry axis \mathcal{A} , which is a set of fixed points of $\xi = \partial/\partial\phi$ and on the event horizon \mathcal{H} . ρ is also a harmonic function on the space of orbits M' of ξ and η . After reductions of the E-M equations for stationary and axisymmetric fields, we obtain a set of equations on the space of orbits M'

$$\nabla_\mu \nabla^\mu \rho = 0, \quad (22)$$

$$-X \nabla_\mu (\rho \nabla^\mu \epsilon) = \rho (\nabla_\mu \epsilon + 2\bar{\psi} \nabla_\mu \psi) \nabla^\mu \epsilon,$$

$$-X \nabla_\mu (\rho \nabla^\mu \psi) = \rho (\nabla_\mu \epsilon + 2\bar{\psi} \nabla_\mu \psi) \nabla^\mu \psi, \quad (23)$$

where ∇_μ is the covariant derivative with respect to the metric $g_{\mu\nu}$ on the space of orbits M' . The black hole solutions are solutions to (22) and (23) for the Carter boundary conditions (Carter, 1973, 1979),^{6,7} which can be written explicitly once we fix a coordinate system on $(M', g_{\mu\nu})$. Once we know the

topology of the event horizon of stationary black holes is $R^1 \times S^2$ (Hawking, 1972, 1973)^{16,17} and the topology of the domain of outer communications is $R^2 \times S^2$ we can show, using Morse theory, that the harmonic function ρ does not have *critical* points on \overline{M}' . It means that a gradient of ρ and its harmonic conjugate function z is non-zero on \overline{M}' (Carter, 1973, 1979).^{6,7} From the asymptotic flatness condition, we have that ρ at large distances behaves as the Weyl canonical cylindrical coordinate. The size of the event horizon fixes the overall scale of ρ . We can take ρ and z , its harmonic conjugate function, as the globally well behaved coordinates on \overline{M}'

$$\rho^2 = c^2 (x^2 - 1) (1 - y^2), \quad z = cxy, \quad (24)$$

where

$$c^2 = M^2 - (J/M)^2 - Q^2.$$

$x = 1$ defines the location of the event horizon \mathcal{H} , $y = \pm 1$ are the two branches of the symmetry axis \mathcal{A} , and we reach the spatial infinity at $x \rightarrow +\infty$. The black hole's boundary conditions are parametrized by three parameters: the angular momentum J , the electric charge Q and the parameter c or the total mass of a black hole. The black hole solution is described by the asymptotically flat spacetime which is regular on the axis \mathcal{A} and on the event horizon \mathcal{H} . These conditions can be translated into the formalism discussed above. The asymptotic boundary conditions are:

$$E = Qy + O(x^{-1}),$$

$$B = O(x^{-1}),$$

$$Y = 2Jy(3 - y^2) + O(x^{-1}),$$

$$X = c^2 x^2 (1 - y^2) + O(x^{-1}), \quad (25)$$

as $x \rightarrow \infty$. The symmetry axis boundary conditions are: E , B , X and Y , should be well behaved functions of x and y which satisfy

$$\partial_x E = O(1 - y^2), \quad \partial_y E = O(1),$$

$$\partial_x B = O(1 - y^2), \quad \partial_y B = O(1),$$

$$\partial_x Y = O(1 - y^2), \quad \partial_y Y + 2(E\partial_y B - B\partial_y E) = O(1 - y^2),$$

$$X = O(1 - y^2), \quad X^{-1}\partial_y X = -2y(1 - y^2)^2 + O(1 - y^2), \quad (26)$$

as $y \rightarrow \pm 1$. The event horizon boundary conditions demand only that E , B , X and Y are well behaved functions of x and y as $x \rightarrow 1$

$$\partial_x E = O(1), \quad \partial_y E = O(1),$$

$$\partial_x B = O(1), \quad \partial_y B = O(1),$$

$$\partial_x Y = O(1), \quad \partial_y Y = O(1),$$

$$X = O(1), \quad X^{-1} = O(1), \quad (27)$$

It was conjectured that the only stationary and axisymmetric black hole solutions satisfying these boundary conditions are described by the Kerr-Newman et al. (Kerr, 1963; Newman et al., 1965)^{22,30} family of solutions satisfying the condition of $c^2 = M^2 - (J/M)^2 - Q^2 > 0$. Great progress in proving correctness of this conjecture was achieved initially in the early seventies (Carter, 1971, 1973; Robinson, 1974)^{5,6,36} and then it culminated in a remarkable black hole uniqueness theorem for vacuum black holes (Robinson, 1975).³⁷

The proof of *no hair theorems* of Carter and Robinson and Robinson's *uniqueness theorem* for the Kerr black hole made use of remarkable divergence identities. The common ancestor of these mysterious identities is the Green identity for the Laplace equation, which is usually applied to show that the Dirichlet or Neuman boundary value problem is well posed. The *no hair theorems* of Carter and Robinson are basically statements about the absence of nontrivial bifurcations for solutions of the harmonic map equations

(22) and (23) with the black hole boundary conditions (25), (26) and (27). It means that if we consider the linearized harmonic map equations for small perturbations of a harmonic map around a black hole solution with fixed values of M , J and Q then there are no perturbations satisfying linearized black hole boundary conditions. The conclusion one could draw from these results (Carter, 1973)⁶ was that if there were other solutions, they should form a disjoint family of solutions parametrized also by three conserved quantities: the total mass M , the total angular momentum J and the electric charge Q .

What remained to be done was to exclude the possibility of other than the Kerr-Newman et al. families of black hole solutions, proving, therefore, *the black hole uniqueness theorem*. For vacuum black holes, this goal was achieved by D. C. Robinson in 1975 who found a nonlinear version of Carter's divergence identity and applied it to the proof of the black hole uniqueness theorem for the Kerr black hole. The most general form of the black hole uniqueness theorem was proven only recently

Theorem 2 Black Hole Uniqueness Theorem (Mazur, 1982a,b; Bunting, 1983) *The only possible stationary and axisymmetric black hole solutions of the Einstein-Maxwell equations satisfying boundary conditions (25), (26) and (27) are the Kerr-Newman et al. solutions subject to the constraint $M^2 - (J/M)^2 - Q^2 > 0$. Black holes are completely characterized by three parameters only: the total mass M , the total angular momentum J and the total electric charge Q . If there is a magnetic charge P in Nature, then black holes will be described completely by the four parameter family of Kerr-Newman et al. solutions with $M^2 - (J/M)^2 - Q^2 - P^2 > 0$.*

Before proceeding with the proof of this *uniqueness theorem for black holes*, I would like first to introduce very useful techniques of the harmonic map theory. The remarkable divergence identities of Carter and Robinson will emerge as a natural consequence of the harmonic map property of the Einstein-Maxwell equations for stationary and axisymmetric fields.

IV. Divergence Identities for Harmonic Maps and No Hair Theorems

Harmonic maps enjoy a lot of nice properties; especially we can say this for harmonic maps between two Riemannian manifolds where the image manifold has non-positive sectional curvature (Yau, 1982).⁴¹ It is well known that if N has non-positive curvature, then every map from M to N is homotopic to a harmonic map with minimal ‘energy’. This existence theorem for harmonic maps between two compact Riemann manifolds was established by Eells and Sampson (1964).⁹ Moreover, there exists one to one and only one harmonic map in each homotopy class of maps between two compact Riemann manifolds M and N , when N has negative curvature and if the image of the first homotopy group $\pi_1(M)$ of M in $\pi_1(N)$ is not a cyclic group (Hartman, 1967).¹⁵ Mathematically, the black hole uniqueness theorem is an example of a result which shows that under certain boundary conditions the same uniqueness theorem for harmonic maps can be extended to the non-compact case. I will present here results discussed first in (Mazur, 1984a,b; Mazur TPUJ 83 preprint)^{26,27} (and independently by Bunting (1983)⁴). Because the *no hair theorems* are concerned with small perturbations of black hole solutions, we will be concerned here with the question of ‘stability’ of harmonic maps to a negatively curved Riemann manifold N . The Euler-Lagrange equations for harmonic maps can be derived from the variational principle (20)

$$\nabla_a \nabla^a \Phi^A + \Gamma_{BC}^A \nabla_a \Phi^B \nabla^a \Phi^C = 0 \quad (28)$$

For stationary and axisymmetric E-M fields eq.(28) is modified in the way that a covariant Laplacian in (28) is replaced by $\rho^{-1} \nabla (\rho \nabla)$, where ρ is a non-negative harmonic function. This modification does not change conclusions we will draw about properties of harmonic maps, because all divergence identities presented below will be multiplied by a non-negative function ρ .

In order to study the problem of bifurcations off a family of black hole solutions we need the linearized form of (28). Let us consider a one-parameter family of maps $f_\tau : M \rightarrow N$, represented locally by functions $\Phi^A(\tau, x)$. A tangent vector $\partial/\partial\tau$ to a curve f_τ with components, $X^A = \partial\Phi^A/\partial\tau$ satisfies the ‘small perturbations’ equation (the Jacobi ‘geodesic deviation’ equation; see e. g. (Misner, 1978)²⁹)

$$D_a D^a X^A + R_{BCD}^A \nabla_a \Phi^B \nabla^a \Phi^C X^D = 0, \quad (29)$$

where

$$D_a X^A = \nabla_a X^A + \Gamma_{BC}^A \nabla_a \Phi^B X^C. \quad (30)$$

It is a general property of nonlinear systems that the nonexistence of linearized perturbations for a given boundary value problem is sufficient to exclude the possibility of a corresponding bifurcating family of exact solutions. The following divergence identity is useful to determine the possibility of bifurcation of solutions to (28)

$$\nabla^a (H_{AB} X^A D_a X^B) = H_{AB} D_a X^A D^a X^B + R_{ABCD} X^A \nabla_a \Phi^B X^C \nabla^a \Phi^D. \quad (31)$$

The divergence term on the l.h.s. of (31), when integrated over M gives the surface term contribution which vanishes for a large class of boundary conditions (when M has a boundary ∂M ; for M compact it vanishes identically). The first term on the r.h.s. of (31) is non-negative when M and N are Riemannian. The second term is also non-negative only when the sectional curvature of N is non-positive. If the surface term vanishes, as happens for black hole solutions, then it follows that each term on the r.h.s. of (31) vanishes separately, i. e.,

$$D_a X^A = 0$$

and

$$X^{[A} \nabla_a \Phi^{B]} = 0.$$

Then from (31) it follows that

$$\Lambda_B^A X^B = 0,$$

where

$$\Lambda_B^A = R_{BCD}^A \nabla_a \Phi^C \nabla^a \Phi^D.$$

There can exist a non-zero solution to the zero eigen-value problem for the matrix Λ if its determinant is zero. In general, a given solution of (28) for which we are seeking bifurcations has the property that Λ is a non-degenerate matrix. One can also see that the number of bifurcations is equal to $\dim(N) -$

$rank(\Lambda)$. We conclude that, in general, there are no bifurcations for harmonic maps when the sectional curvature of N is non-positive (unless we have a degenerate case). The Carter and Robinson *no hair theorems* are simply very special cases of the result I have described briefly above. The positive definite divergence identity (31) when applied to electrovacuum black holes, i. e., to solutions of (22) and (23) satisfying black hole boundary conditions (25), (26) and (27) gives as a result *the black hole no hair theorem* (Carter, 1971, 1973; Robinson, 1974; Mazur, 1984a,b).^{5,6,36,26,?} This is so because the image space of the Ernst potentials ϵ and ψ is a Riemannian space with negative sectional curvature (Mazur, 1982a,b, 1984a,b).^{23,24,26,27} It is quite interesting that the complicated divergence identities of Carter and Robinson are just special cases of the general divergence identity (31) for perturbations of harmonic maps to negatively curved Riemann manifold N . It is obvious that using a particular Ernst's coordinate system on (N, H_{AB}) , as given by (21), one can obtain a rather unenlightening form of the identity (31). It is the advantage of the geometrical approach of harmonic map theory which could bring us a better understanding of global black hole solutions. We have, therefore, a nice theorem

Theorem 3 No Hair Theorem (*Carter, 1971, Robinson, 1974*) *There do not exist regular, small perturbations of the; stationary and axisymmetric black hole solutions of the E-M equations preserving boundary conditions (25), (26) and (27) for fixed values of the mass M , the angular momentum J and the electric charge Q of a black hole. The only deformations of black holes that do exist are those obtained by a change of M , J , and Q .*

It is also easy to see that starting with the divergence identity for small perturbations of harmonic maps one can obtain the *global* divergence identity with the required positivity property when N has a non-positive sectional curvature. Consider a curve of harmonic maps f_τ , $\tau \in [0, 1]$. Then f_0 and f_1 are homotopically equivalent. For each the r.h.s. of (31) is non-negative. If we integrate both sides of the divergence identity (31) along a geodesic curve f_τ the right hand side (r.h.s.) will be non-negative. The left hand side (l.h.s.) will have a form

$$\int_0^1 d\tau \nabla_a \nabla^a (H_{AB} X^A X^B) = \nabla_a \nabla^a \int_0^1 d\tau (\partial/\partial\tau, \partial/\partial\tau) \geq 0. \quad (32)$$

Taking advantage of the fact that there always exists a unique geodesic joining two points f_0 and f_1 on a simply connected Riemannian manifold N with a negative sectional curvature one can evaluate the integral (32). Because f_τ is a geodesic, it means that the norm $(\partial/\partial\tau, \partial/\partial\tau)$ is a constant along f_τ , i. e., $(\partial/\partial\tau, \partial/\partial\tau) = c^2 = \text{constant}$. One can easily see that the geodesic distance between f_0 and f_1 is:

$$S(f_0, f_1) = \int_0^1 d\tau \sqrt{(\partial/\partial\tau, \partial/\partial\tau)} = c \int_0^1 d\tau = c.$$

It means that the τ integral in (32) is equal to $S^2(f_0, f_1)$. In this way we have arrived at the Bunting divergence inequality (Bunting, 1983)⁴

$$\nabla^a \nabla_a S^2(f_0, f_1) \geq 0. \quad (33)$$

A similar identity where S^2 was replaced by an arbitrary function of S^2 with positive first and second derivatives was proposed also by the present author (Mazur, 1984a,b; Mazur TPJU-22/83 preprint).^{26,27} One can apply the divergence identity (33) to the space N of Ernst's potentials. N is Riemannian and it has negative sectional curvature. The 'distance' function $S(f_0, f_1)$ for two solutions of the harmonic map equations (28) (and in the special case of Ernst equations (23)) satisfying the same boundary conditions is non-negative and vanishes only on the boundary. For the case of Ernst's equations for the stationary and axisymmetric fields the divergence identity is modified by the presence of the harmonic, positive function ρ :

$$\nabla_\mu (\rho \nabla^\mu S^2(f_0, f_1)) \geq 0.$$

When we integrate this identity over the manifold of orbits M' we obtain surface terms which vanish. The contribution from the symmetry axis \mathcal{A} and the event horizon \mathcal{H} vanishes in virtue of the boundary conditions (26) and (27) and because ρ is vanishing there. The asymptotic boundary conditions (25) imply vanishing of a surface integral at infinity where $\rho \rightarrow \infty$. The r.h.s. of the identity (33) vanishes only when $S(f_0, f_1)$ is constant everywhere. But $S(f_0, f_1)$ vanishes on the boundary, so it must vanish everywhere. If we take f_0 to be the Kerr-Newman et al. black hole solution and for f_1 we take another possible black hole solution, then vanishing of $S(f_0, f_1)$ implies $f_0 \equiv f_1$, i. e., *the black hole uniqueness theorem* (Mazur, 1982a,b; Bunting,

1983).^{23,24,4} We have seen that the *no hair theorems* and the *black hole uniqueness theorem* can be proved applying a generalization of the Green divergence identity to the case of harmonic maps to Riemannian manifolds with non-positive sectional curvature. As it was mentioned above, one can easily generalize the inequality (33) by taking an even function of S with positive first and second derivatives with respect to S . Consider a function of $\sigma = S^2$, $h(\sigma)$, and evaluate its Laplacian

$$\nabla^2 h(\sigma) = h'(\sigma) \nabla^2 \sigma + h''(\sigma) (\nabla \sigma)^2.$$

Then

$$\nabla^2 h(\sigma) \geq 0,$$

if $h'(\sigma) > 0$, $h''(\sigma) > 0$ and (33) is satisfied, i. e., if N has non-positive sectional curvature (Mazur, 1982b, 1983, 1984a,b).^{24,26,27}

The method the present author used in his proof of *the black hole uniqueness theorem* is based on a divergence identity for the Ernst equations (23), whose derivation was based on the observation that the image manifold of Ernst's potentials is a homogeneous, symmetric Kähler manifold $N = SU(2, 1)/S(U(2) \times U(1))$. This method seems to be much simpler in application because for the electrovacuum black holes the Riemannian space of Ernst's potentials ϵ and ψ is a symmetric space with the $SU(2, 1)$ isometry group. Because of this large symmetry, one can use group theoretical methods to construct an $SU(2, 1)$ invariant two-point function $h(f_0, f_1)$ which is a certain function of the geodesic distance between two harmonic maps f_0 and f_1 . The algebraic explicit construction of the global divergence identity offers a simple understanding of the otherwise mysterious Robinson identity (Mazur, 1982a,b; Mazur, 1984a,b),^{23,24,26,27} because the identity produced this way reproduces in the vacuum case the Robinson identity (Robinson, 1975).³⁷ The construction I am going to describe here has a much broader range of applications than solely the black hole uniqueness theorems. It can be applied to any nonlinear elliptic sigma model on symmetric Riemannian spaces with non-compact isometry groups, or, which is equivalent, with negative sectional curvature (Mazur, 1984a,b; Mazur and Richter, 1985; Breitenlohner and Maison, 1986).^{26,27,28,3}

V. A Global Identity for Nonlinear Sigma Models and Its Applications: Black Hole Uniqueness Theorem

Before discussing the most general case of sigma models on symmetric Riemannian and hyperbolic spaces and the divergence identity associated with them, let me first demonstrate explicitly that the metric (21) related to the electrovacuum Ernst equations is the left invariant metric on the homogeneous, symmetric and Kahler space $N = SU(2,1)/S(U(2) \times U(1))$. The detailed construction is described in (Mazur, 1982a,b, 1983).^{23,24,25} There are many ways to see this. The metric (21) is an example of a Hermitian metric on an (almost) complex manifold N with complex coordinates z^α (in the case of electrovacuum Ernst equations, $\alpha = 1, 2$, $z^1 = \epsilon$, $z^2 = \psi$):

$$ds^2 = k_{\alpha\bar{\beta}} dz^\alpha d\bar{z}^\beta.$$

A complex Hermitian manifold N is a Kähler manifold if it admits a closed non-degenerate $(1,1)$ form

$$\omega = -\frac{i}{2} k_{\alpha\bar{\beta}} dz^\alpha \wedge d\bar{z}^\beta.$$

One can easily see that the metric (21) is Kähler (Mazur 1983),²⁵ i. e.,

$$\partial_\gamma k_{\alpha\bar{\beta}} = \partial_{\bar{\alpha}} k_{\gamma\bar{\beta}}, \quad \partial_{\bar{\gamma}} k_{\alpha\bar{\beta}} = \partial_{\bar{\beta}} k_{\alpha\bar{\gamma}},$$

where $\partial_\alpha = \partial/\partial z^\alpha$ and $\partial_{\bar{\alpha}} = \partial/\partial \bar{z}^\alpha$. Moreover, the metric (21) has 8 holomorphic Killing vectors which generate the Lie algebra of the pseudounitary group $SU(2,1)$. The group $G = SU(2,1)$ acts on N nonlinearly by holomorphic (homographic) maps. One can see that by introducing new Ernst coordinates w^α on N :

$$z^1 = \frac{w^1 - 1}{w^1 + 1}, \quad z^2 = \frac{w^2}{w^1 + 1}.$$

and showing that the following holomorphic transformations of w^α are invariant transformations of the metric (21)

$$w^{1'} = \frac{u_1^1 w^1 + u_2^1 w^2 + u_3^1}{u_1^3 w^1 + u_2^3 w^2 + u_3^3},$$

$$w^{2'} = \frac{u_1^2 w^1 + u_2^2 w^2 + u_3^2}{u_1^3 w^1 + u_2^3 w^2 + u_3^3},$$

where the matrix u_{β}^{α} , $\alpha, \beta = 1, 2, 3$ satisfies the pseudo-unitarity condition

$$u^+ u = \eta, \quad \eta = \text{diag}(1, 1, -1) \quad (34)$$

The group $G = SU(2, 1)$ acts on N simply transitively. This means that N is a homogeneous Kahler manifold. Every homogeneous manifold of G is diffeomorphic to the left coset space G/H for some subgroup H of G . In our case, it can be easily seen that $H = S(U(2) \times U(1))$. Fix a point $p_0 \in N$, such that $w^{\alpha}(p_0) = 0$. Every point $p \in N$ can be reached from p_0 by a map $u : p = up_0$, $u \in G$. Two maps $u, u' \in G$ represent the same point if $u \sim u' = uh$, $h \in H$. H is the isotropy subgroup of p_0 which is isomorphic to $S(U(2) \times U(1))$. The equivalence relation \sim defines the coset space G/H . The homogeneous space $N = SU(2, 1)/S(U(2) \times U(1))$ is also a symmetric space. The last property helps to construct a unique left G invariant metric on N in terms of a coset representative g .

We would like to reformulate the Ernst equations (23) associated with the metric (21) in terms of group theoretical objects, like a coset representative $g(uH) = g(u)$, making their $SU(2, 1)$ covariance explicit. This will help us to construct the generalized Robinson identity in the obvious way. Exploiting the $SU(2, 1)$ covariance of the field equations (23) one can obtain an $SU(2, 1)$ invariant divergence identity for two solutions of (23). The metric (21) is a left $SU(2, 1)$ -invariant metric on a symmetric space $N = SU(2, 1)/S(U(2) \times U(1))$. This leads to a nonlinear sigma model form of the Ernst equations (23), once we go from the Ernst parametrization of N to a coset space representation of N . We give here a short derivation of a global divergence identity for nonlinear sigma models on symmetric spaces and apply it to the electrovacuum Ernst equations.

We define the symmetric space as a triple (G, H, μ) where G is a connected Lie group, and H is a closed subgroup of G defined by an involutive automorphism μ of G such that $(G_{\mu})_0 \subset H \subset G_{\mu}$ with G_{μ} and $(G_{\mu})_0$ being the set of fixed points of μ and its identity component, respectively. An involutive automorphism μ defines also a smooth mapping g of a coset space G/H into G :

$$g(u) = u\mu(u)^{-1}, \quad g(uH) = g(u).$$

The G -valued field g satisfies a constraint:

$$g\mu(g) = I.$$

In terms of g one can naturally introduce the left G -invariant metric on G/H :

$$dS^2 = \frac{1}{2}Tr(dgg^{-1})^2.$$

If G is a non-compact group and H is its maximally compact subgroup, then the metric $\frac{1}{2}Tr(dgg^{-1})^2$ is Riemannian. A harmonic map from a Riemannian space M to a Riemannian coset space $N = G/H$ is called a *nonlinear sigma model* with the Lagrangian

$$L = \frac{1}{2}Tr(J_a J^a), \quad J_a = \nabla_a g g^{-1}, \quad (35)$$

and field equations

$$\nabla_a J^a = 0. \quad (36)$$

The field equation (36) is covariant under the left G translation on G/H . A global divergence identity for the nonlinear sigma model (35) can be obtained exactly in the same way as the Green identity for the Laplace equation. To this end, consider two fields g_0 and g_1 , not necessarily solutions of (36). Define a field

$$\Phi = g_0 g_1^{-1},$$

which transforms under the rigid left G -translations on N ,

$$u_0 \rightarrow uu_0, u_1 \rightarrow uu_1,$$

$u \in G$, in a simple way:

$$\Phi \rightarrow u\Phi u^{-1}.$$

One can easily see that the trace of Φ is invariant under the rigid G -translations of two points on N . It means that $Tr\Phi$ is a function of geodesic distance between g_0 and g_1 :

$$Tr\Phi = h(S), \quad S = S(g_0, g_1).$$

Evaluating the covariant Laplacian of $Tr\Phi$ and using (35) we arrive at the generalized Green identity for nonlinear sigma models (Mazur, 1982a,b, 1984a,b)^{23,24,26,27}

$$\nabla_a \nabla^a h = Tr [\Phi \nabla_a (J^{(0)a} - J^{(1)a})] + Tr [\Phi (J_a^{(0)} J^{(0)a} + J_a^{(1)} J^{(1)a} - 2J_a^{(0)} J^{(1)a})] \quad (37)$$

where $J_a^{(i)} = \nabla_a g_i g_i^{-1}$, $i = 0, 1$.

Theorem 4 (Mazur, 1982b, 1984a) *A global identity (37) for Riemannian nonlinear sigma models has a positive definite right hand side if g_0 and g_1 are solutions to (36) and the following conditions on N are satisfied:*

1. $N = G/H$ is a Riemannian symmetric space with non-compact isometry group G .
2. N has a non-positive sectional curvature.

We can see now a direct correspondence of this result to the basic properties of harmonic mappings to a Riemannian space N with non-positive sectional curvature discussed before. For the electrovacuum Ernst equations, the identity (37) is a generalization of the Robinson identity and also it coincides with Robinson's identity for the vacuum case. This form of the divergence identity (37) proves to be very useful in evaluating surface terms for black hole boundary conditions because the function $h = Tr\Phi$ turns out to be only a rational function of Ernst potentials. The symmetric space $N = SU(2, 1)/S(U(2) \times U(1))$ is singled out by the existence of an inner involutive automorphism of $G = SU(2, 1)$:

$$u \rightarrow \mu(u) = \eta u \eta^{-1}, \quad \eta = (1, 1, -1),$$

because $\mu(H) = H$, $H = S(U(2) \times U(1))$. A coset representative

$$g = u \mu(u)^{-1},$$

which is an element of $G = SU(2, 1)$, i. e.,

$$g^+ \eta g = \eta,$$

satisfies also a constraint

$$g\mu(g) = g\eta g\eta^{-1} = I.$$

This implies that g is a hermitian matrix $g^+ = g$. Writing

$$g = \eta + 2P,$$

one can see that P satisfies the following constraints:

$$P^+ = P, (P\eta)^2 = -P\eta.$$

We will recover the Ernst parametrization of $SU(2,1)/S(U(2) \times U(1))$ and satisfy constraints on P if we write

$$P^{\alpha\beta} = v^\alpha \bar{v}^\beta,$$

where

$$v^\alpha \eta_{\alpha\beta} \bar{v}^\beta = -1, \alpha, \beta = 1, 2, 3,$$

and define

$$w^1 = \frac{v^1}{v^3}, w^2 = \frac{v^2}{v^3}.$$

w^1 and w^2 can be written in terms of Ernst's potentials ϵ and ψ as follows:

$$w^1 = \frac{1 + \epsilon}{1 - \epsilon}, w^2 = \frac{2\psi}{1 + \epsilon}.$$

Using the Ernst parametrization of the coset space element g which was discussed above one can calculate a function $h(S) = Tr\Phi$ in terms of the Ernst potentials X_i, Y_i, E_i and $B_i, i = 0, 1$

$$\begin{aligned} h(S) = Tr\Phi = 3 + X_0^{-1} X_1^{-1} \{ & (X_0 - X_1)^2 + 2(X_0 + X_1) [(E_0 - E_1)^2 + (B_0 - B_1)^2] + \\ & + [(E_0 - E_1)^2 + (B_0 - B_1)^2]^2 + [Y_0 - Y_1 + 2(E_1 B_0 - E_0 B_1)]^2 \} \end{aligned} \quad (38)$$

The black hole uniqueness theorem for electrically charged black holes can be obtained by a straightforward application of the divergence identity (37). When we integrate the l.h.s. of (37) we obtain a surface term which can be shown, after using (38), to be vanishing for the black hole boundary conditions (25), (26) and (27). Because the r.h.s. of (37) is positive definite, it means that $h = \text{const.}$ This is consistent with the boundary conditions and (38) only when $X_0 = X_1$, $Y_0 = Y_1$, $E_0 = E_1$ and $B_0 = B_1$. It means that the Kerr-Newman et al. family of solutions with $M^2 - (J/M)^2 - Q^2 > 0$ characterizes completely the stationary equilibrium black hole states in the Einstein-Maxwell theory. Concluding this talk, I would like to point out that it was only possible to prove black hole uniqueness theorems for the stationary black holes which are also axisymmetric. It is fortunate that the Einstein-Maxwell equations reduce to harmonic mapping equations which are moreover conformally invariant in the case of stationary and axisymmetric black holes. One may hope that the reasonable extension of arguments presented here may lead to the solution of the uniqueness problem for stationary black holes, without assumption of axial symmetry.

Acknowledgments

This work was supported in part by the national Science Foundation through Grant No. PHY 83-18350 with Syracuse University and a travel grant, also. I would like to thank the General Relativity Society for giving me the opportunity to present this material.

The address and affiliation(s) at the time this review was written were the following: Department of Physics, Syracuse University, Syracuse, N.Y. 13244-1130, U. S.A. and Department of Theoretical Physics, Jagellonian University, Kraków, Poland.

References

- ¹Bardeen, J. M., Carter, B., and Hawking, S. W., (1973). “The four laws of black hole mechanics”, *Commun. Math. Phys.*, 31, 161.
- ²Bochner, S., (1940). “Harmonic surfaces in Riemann metric”, *Trans. Amer. Math. Soc.* 47, 146.
- ³Breitenlohner, P. and Maison, D., (1986). “Solitons in Kaluza-Klein theories”, in *Solitons in General Relativity*, eds. H. Morris and R. Dodd, (New York: Plenum).
- ⁴Bunting, G., (1983). “Proof of the uniqueness conjecture for black holes”, Ph. D. Thesis (unpublished) University of New England, Armidale, N. S. W. .
- ⁵Carter, B., (1971). “Axisymmetric black hole has only two degrees of freedom”, *Phys. Rev. Lett.* 26, 331.
- ⁶Carter, B., (1973). “Black hole equilibrium states”, in *Black Holes*, eds. C. de Witt and B. S. De Witt, pp.57-217, New York: Gordon and Breach.
- ⁷Carter, B., (1979). “The general theory of the mechanical, electromagnetic and thermodynamic properties of black holes”, in *An Einstein Centenary Survey*, eds. S. W. Hawking and W. Israel, pp. 294-369, Cambridge, Cambridge University Press.
- ⁸Doroshkevich, A. G., Zel’dovich, Ya. B., and Novikov, I. D., (1966). “Gravitational collapse of non-symmetric and rotating bodies”, *Sov. Phys. J. E. T. P.* 22,122.
- ⁹Eells, J., and Sampson, J. H. (1964). “Harmonic mappings of Riemannian manifolds”, *Amer. J. Math.* 86, 109.
- ¹⁰Ernst, F. J., (1968). “New formulation of the axially symmetric gravitational field problem I, II”, *Phys. Rev.* 167, 1175; *ibid.* 168, 1415.
- ¹¹Fuller, F. B., (1954). “Harmonic mappings”, *Proc. Nat. Acad. Sci. USA* 40, 987.
- ¹²Geroch, R. P., (1971). “A method for generating solutions of Einstein’s equations”, *J. Math. Phys.* 12, 918.

- ¹³Hajicek, P., (1973). “General theory of vacuum ergospheres”, Phys. Rev. D7, 2311.
- ¹⁴Hartle, J. B., and Hawking, S. W., (1973). “Solutions of the Einstein-Maxwell equations with many black holes”, Commun. Math. Phys. 26, 87.
- ¹⁵Hartman, P., (1967). “On homotopic harmonic maps”, Canada J. Math. 19, 673.
- ¹⁶Hawking, S. W., (1972). “Black holes in general relativity”, Commun. Math. Phys. 25, 152.
- ¹⁷Hawking, S. W., (1973). “The event horizon”, in Black Holes, eds. C. De Witt and B. S. De Witt, pp. 5-34, New York, Gordon and Breach.
- ¹⁸Hawking, S. W., and Ellis, G. F. R., (1973). “The large scale structure of spacetime”, cambridge, Cambridge University Press.
- ¹⁹Ipsen, J., (1971). “Electromagnetic test fields around a Kerr-Metric black hole”, Phys. Rev. Lett. 27, 529.
- ²⁰Israel, W., (1967). “Event horizons in static vacuum spacetimes”, Phys. Rev. 164, 1776.
- ²¹Israel, W., (1968). “Event horizons in static electrovac spacetimes”, Commun. Math. Phys. 8, 245.
- ²²Kerr, R. P., (1963). “Gravitational field of a spinning mass as an example of algebraically special metrics”, Phys. Rev. Lett. 11, 237.
- ²³Mazur, P. O., (1982a). “Proof of uniqueness of the Kerr-Newman black hole solution”, J. Phys. A15, 3173.
- ²⁴Mazur, P. O., (1982b). “Properties and integrability of the Ernst equations”, Ph. D. Thesis, Jagellonian University, Krakow, Poland, unpublished (in Polish).
- ²⁵Mazur, P. O., (1983). “A relationship between the electrovacuum Ernst equations and nonlinear sigma model”, Acta Phys. Polon. B14, 219.

- ²⁶Mazur, P. O., (1984a). “A global identity for nonlinear sigma models”, Phys. Lett. 100A, 341; Jagellonian University 1983 preprint TPJU-22/83.
- ²⁷Mazur, P. O., (1984b). “Black hole uniqueness from a hidden symmetry of Einstein’s gravity”, GRG16, 211.
- ²⁸Mazur, P. O. and Richter, E., (1985). “Harmonic maps and uniqueness of axisymmetric monopole solutions”, Phys. Lett. 109A, 429.
- ²⁹Misner, C. W., (1978). “harmonic maps as models for physical theories”, Phys. Rev. D18, 4510.
- ³⁰Newman, E. T., Couch, E., Chinnapared, K., Exton, A., Prakash, A., and Torrence, R., (1965). “Metric of a rotating charged mass”, J. Math. Phys. 6, 918.
- ³¹Oppenheimer, J. R. and Snyder, H., (1939). “On continued gravitational contraction”, Phys. Rev. 56, 455.
- ³²Penrose, R., (1969). “Gravitational collapse: the role of general relativity”, Riv. del Nuovo Cimento 1, 252.
- ³³Price, R. H., (1972). “Nonspherical perturbations of relativistic gravitational collapse. I. Scalar and gravitational perturbations”, Phys. Rev. D5, 2419.
- ³⁴Press, W. H. and Teukolsky, S. A., (1973). “Perturbations of rotating black hole. II. Dynamical stability of the Kerr metric”, Phys. Rev. D5, 2439.
- ³⁵Regge, T. and Wheeler, J. A., (1957). “Stability of a Schwarzschild singularity”, Phys. Rev. 108, 1063.
- ³⁶Robinson, D. C., (1974). “Classification of black holes with electromagnetic fields”, Phys. Rev. D10, 458.
- ³⁷Robinson, D. C., (1975). “Uniqueness of the Kerr black hole”, Phys. Rev. Lett. 34, 905.
- ³⁸Wald, R. M., (1972). “Nonspherical gravitational collapse and black hole uniqueness”, Ph. D. Thesis, Princeton University, unpublished.

- ³⁹Wald, R. M., (1973). “On perturbations of a Kerr black hole”, J. Math. Phys. 14, 1453.
- ⁴⁰Vishveshwara, C. V., (1970). “Stability of the Schwarzschild metric”, Phys. Rev. D1, 3870.
- ⁴¹Yau, S. T., (1982). “Survey on partial differential equations in differential geometry”, in Seminar on Differential Geometry, ed. Shing-Tung Yau, Princeton, Princeton University Press.